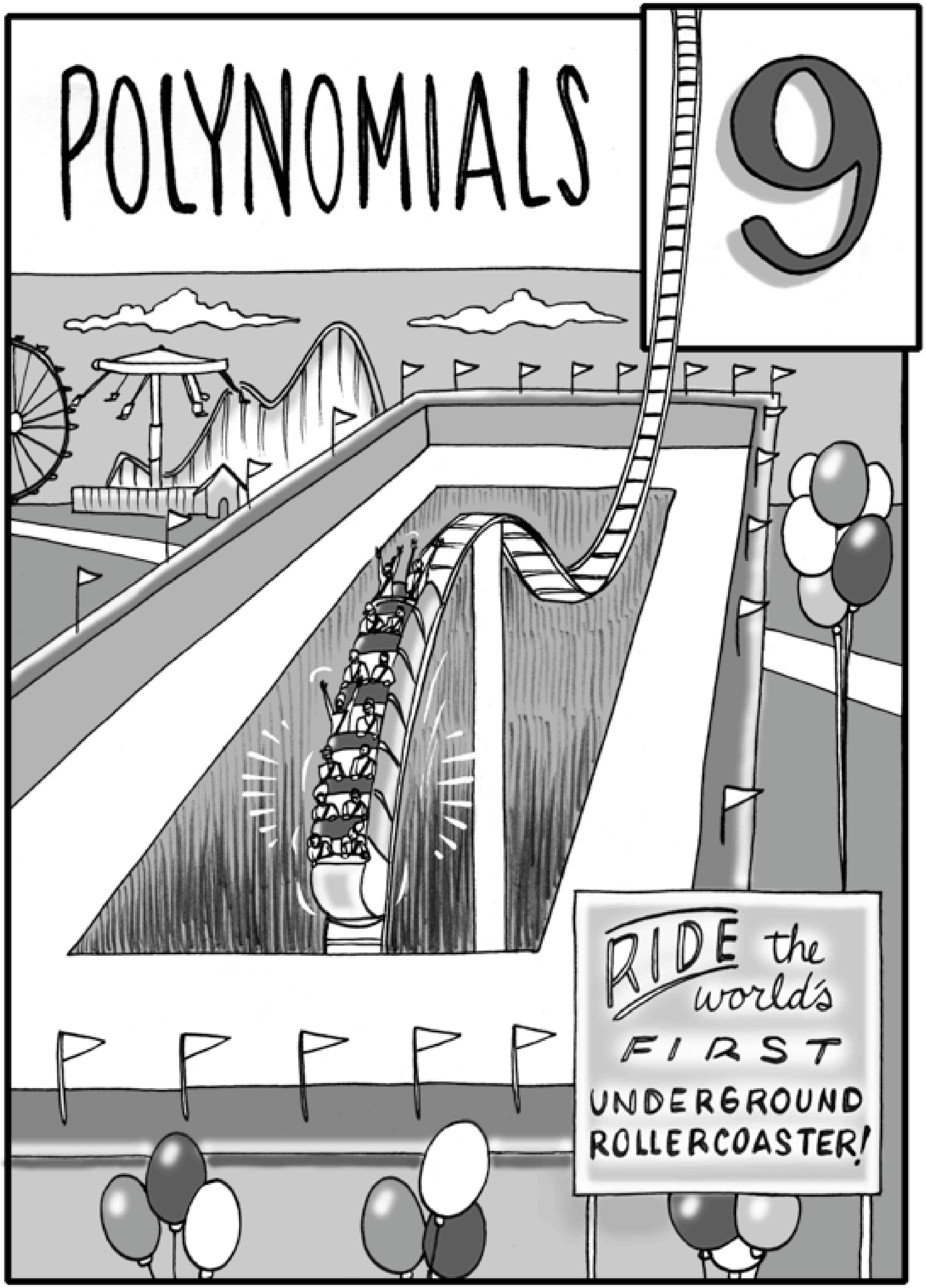


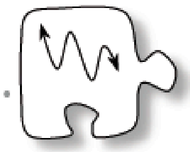
POLYNOMIALS
9
RIDE the world's FIRST UNDERGROUND ROLLERCOASTER!

Chapter 9 Teacher Guide

Section	Lesson	Days	Lesson Objectives	Materials	Homework
9.1	9.1.1	2	Sketching Graphs of Polynomial Functions	• Poster graph paper • Colored poster pens	9-8 to 9-17 and 9-18 to 9-27
	9.1.2	1	More Graphs of Polynomials	None	9-38 to 9-46
	9.1.3	1	Stretch Coefficients for Polynomial Functions	None	9-56 to 9-64
9.2	9.2.1	1	Introducing Imaginary Numbers	None	9-72 to 9-80
	9.2.2	1	Complex Roots	None	9-89 to 9-97
	9.2.3	1	More Complex Numbers and Equations	• Lesson 9.2.3 Res. Pg. (optional) • Scissors (optional)	9-105 to 9-113
9.3	9.3.1	1	Polynomial Division	None	9-121 to 9-130
	9.3.2	2 or 3	Factors and Integral Roots	None	9-140 to 9-147 and 9-148 to 9-155 and 9-156 to 9-164
	9.3.3	1	An Application of Polynomials	• Graph paper • Scissors • Tape	9-169 to 9-176
Chapter Closure		Varied Format Options			

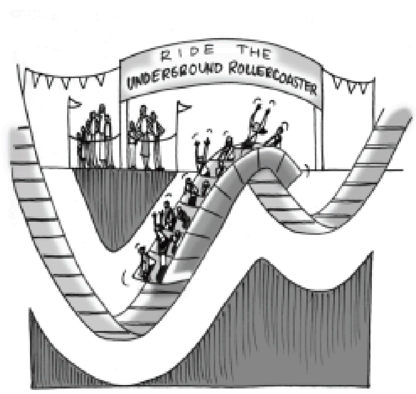
Total: **11 or 12 days plus optional time for Chapter Closure**

9.1.1 How can I predict the graph?



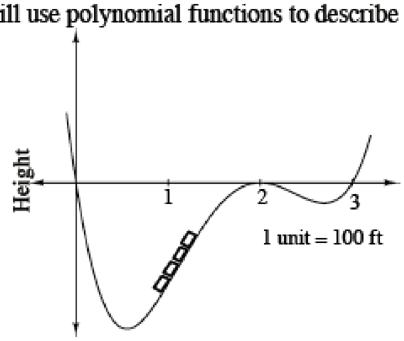
Sketching Graphs of Polynomial Functions

In previous courses and chapters, you learned how to graph many types of functions, including lines and parabolas. Today you will work with your team to apply what you know to more complicated **polynomial** equations. Just as quadratic polynomial equations can be written in standard or factored form, other polynomial equations can be written in standard or factored form. For example, $y = x^4 - 4x^3 - 3x^2 + 10x + 8$ is in standard form, and it can be written in factored form as $y = (x+1)^2(x-2)(x-4)$.



During this lesson, you will develop techniques for sketching the graph of a polynomial function from its equation, and you will **justify** why those techniques work.

9-1. The Mathamericaland Carnival Company has decided to build a new roller coaster to use at this year's county fair. The new coaster will have a very special feature: part of the ride will be underground. The designers will use polynomial functions to describe different pieces of the track. Part of the design is shown at right. Your task is to guess a possible equation to represent the track and test it on your graphing calculator. To help get an idea of what to try, start by checking the graphs of the equations given below. Think about how the graphs are the same and how they are different.



$y = x(x-2)$

$y = (x-2)^2$

$y = x(x-2)(x-3)$

Your task: Use the information you found by graphing the above equations to help you make guesses about the equation that would produce the graph of the roller coaster. Once you have found a graph that has a shape close to this one, try zooming in or changing the viewing window on your graphing calculator to see the details better. Keep track of what you tried and the equations you find that fit most accurately.

9-2. POLYNOMIAL FUNCTION INVESTIGATION

In this **investigation**, you will determine which information in a polynomial equation can help you sketch its graph.

Your task: With your team, create summary statements explaining the relationship between a polynomial equation and its graph. To accomplish this task, first divide up the equations listed below so each team member is responsible for two or three of them. Make a complete graph of each of your functions. Whenever possible, start by making a sketch of your graph without using a graphing calculator. Then, as a team, share your observations including your responses to the "Discussion Points." Choose two or three equations that can be used to represent all of your findings. You can choose them from the list below, or you can create new ones as a team.

The form of your presentation to the class can be on a poster, transparencies or as a PowerPoint™ presentation. Whichever format your teacher decides, make sure you include complete graphs and summary statements that are well **justified**.

$P_1(x)=(x-2)(x+5)^2$ $\quad$ $P_2(x)=2(x-2)(x+2)(x-3)$

$P_3(x)=x^4-21x^2+20x$ $\quad$ $P_4(x)=(x+3)^2(x+1)(x-1)(x-5)$

$P_5(x)=-0.1x(x+4)^3$ $\quad$ $P_6(x)=x^4-9x^2$

$P_7(x)=0.2x(x+1)(x-3)(x+4)$ $\quad$ $P_8(x)=x^4-4x^3-3x^2+10x+8$

Discussion Points

What can we predict from looking at the equation of a polynomial?
Why does this make sense?

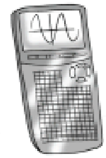
Which form of a polynomial equation is most useful for making a graph?
What information does it give?

How can we use the equation to help predict what a useful window might be?

Which examples are most helpful in finding the connections between the equation and the graph?

How does changing the exponent on one of the factors change the graph?

Further Guidance



9-3. As a team, examine the first polynomial $P_1(x) = (x-2)(x+5)^2$.

a. What family of functions is it a member of? How do you know? Based on its equation, sketch the shape of its graph.

b. Now use your graphing calculator to graph $P_1(x)$. Label the x-intercepts. How are the x-intercepts related to the equation? "Reading" from left to right along the x-axis, describe the graph before the first x-intercept, between x-intercepts, and after the last x-intercept.

9-4. Continuing as a team, examine the equation $P_2(x) = 2(x-2)(x+2)(x-3)$.

a. How many distinct (different) factors are there? How many x-intercepts would you predict for its graph? Draw the graph and label the x-intercepts. How is this graph similar to or different from the graph of $P_1(x)$?

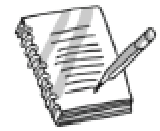
b. Does the factor 2 have any effect on the x-intercepts? On the shape of the graph? On the y-intercepts? How would the graph change if the factor 2 were changed to be a factor -2?

9-5 What is different about $P_3(x) = x^4 - 21x^2 + 20x$? What x-intercept(s) can you determine from the equation, before graphing with the calculator? Explain how you know. Use the graph to figure out exactly what the other intercepts are. Explain how you can prove that your answers are exact.

9-6. With your team, divide up the work to **investigate** $P_4(x)$ through $P_8(x)$ and continue your **investigation**, referring back to the "your task" statement and the discussion points in problem 9-2.

Further Guidance section ends here.

9-7. Based on what your team learned and on the class discussion, record your own list of useful **strategies** for graphing polynomial functions. Use as many of the new vocabulary words as you can and write down the ones you are not sure of yet. You will add to and refine this list over the next several lessons.



May 01, 2011

Review & Preview

9-8. For each equation below, make tables that include x-values from –2 to 2 and draw each graph.

a. $y=(x-1)^2(x+1)$

b. $y=(x-1)^2(x+1)^2$

c. $y=x^3-4x$

d. What are the parent functions for these equations?

9-9. **Polynomials** are expressions that can be written as a sum of terms of the form:

(any number) $\cdot\ x^{\text{(whole number)}}$

Which of the following equations are polynomial equations? For those that are not polynomials, explain why not. Check the lesson 9.1.1 Math Notes box for further details about polynomials.

a. $f(x)=8x^5+x^2+6.5x^4+6$

b. $y=\frac{3}{5}x^6+19x^2$

c. $y=2^x+8$

d. $f(x)=9+\sqrt{x}-3$

e. $P(x)=7(x-3)(x+5)^2$

f. $y=x^2+\frac{1}{x^2+5}$

g. Write an equation for a new polynomial function and then write an equation for a new function that is not a polynomial.

9-10. Describe the possible numbers of intersections for each of the following pairs of graphs. Sketch a graph for each possibility. For example, a circle could intersect a line twice, once, or not at all. Your solution to each part should include all of the possibilities and a sketched example of each one.

a. Two different lines.

b. A line and a parabola.

c. Two different parabolas.

d. A parabola and a circle.

9-11. Solve the following system: $y=x^2-5$
$y=x+1$

9-12. A table can be used as a useful tool for finding some inverse functions. When the function has only one x in it, the function can be described with a sequence of operations, each applied to the previous result. Consider the following table for $f(x)=2\sqrt{x-1}+3$.

	1st	2nd	3rd	4th
What f does to x:	subtracts 1	$\sqrt{\ }$	multiplies by 2	adds 3

Since the inverse must undo these operations, in the opposite order, the table for $f^{-1}(x)$ would look like the one below.

	1st	2nd	3rd	4th
What does f^{-1} to x:	subtracts 3	divides by 2	$(\ )^2$	adds 1

a. Copy and complete the following table for $g^{-1}(x)$ if $g(x)=\frac{1}{3}(x+1)^2-2$

	1st	2nd	3rd	4th
What g does to x:	adds 1	$(\ )^2$	divides by 3	subtracts 2
What g^{-1} does to x:				

b. Write the equations for $f^{-1}(x)$ and $g^{-1}(x)$.

9-13. Describe the difference between the graphs of $y=x^3-x$ and $y=x^3-x+5$.

9-14. Solve the equations below.

a. $\frac{3x}{x+2}+\frac{7}{x-2}=3$

b. $\frac{x-7}{x-5}=\frac{6}{x}$

9-15. An arithmetic sequence starts out –23, –19, –15 …

a. What is the rule?

b. How many times must the generator be applied so that the result is greater than 10,000?

9-16. Artemis was putting up the sign at the County Fair Theater for the movie "ELVIS RETURNS FROM MARS." He got all of the letters he would need and put them in a box. He reached into the box and pulled out a letter at random.

a. What is the probability that he got the first letter he needed when he reached into the box?

b. Once he put the first letter up, what is the probability that he got the second letter he needed when he reached into the box?

9-17. Without a calculator, find two solutions $0^\circ \le \theta < 360^\circ$ that make each of the following equations true.

a. $\cos\theta=\frac{1}{2}$

b. $\tan\theta=-\frac{\sqrt{2}}{2}$

c. $\sin\theta=\frac{\sqrt{3}}{2}$

d. $\cos\theta=-\frac{\sqrt{3}}{2}$

9-18. Which of the following equations are polynomial functions? For each one that is not, **justify** why not.

a. $y = 3x^2 + 2x^2 + x$ b. $y = (x-1)^2(x-2)^2$

c. $y = x^2 + 2^x$ d. $y = 3x - 1$

e. $y = (x-2)^2 - 1$ f. $y^2 = (x-2)^2 - 1$

g. $y = \frac{1}{x^2} + \frac{1}{x} + \frac{1}{2}$ h. $y = \frac{1}{2}x + \frac{1}{3}$

i. $y = x$ j. $y = -7$

9-19. Samantha thinks that the equation $(x-4)^2 + (y-3)^2 = 25$ is equivalent to the equation $(x-4) + (y-3) = 5$. Is she correct? Are the two equations equivalent? Explain how you know. If they are not equivalent, explain Samantha's mistake.

9-20. Find the **roots** (the solutions when $y = 0$) of each of the following polynomial functions.

a. $y = x^2 - 6x + 8$ b. $f(x) = x^2 - 6x + 9$

c. $y = x^3 - 4x$

9-21. Sketch a graph of $y = x^2 - 7$.

a. How many roots does this graph have?

b. What are the roots of the function?

9-22. Solve $x^2 + 2x - 5 = 0$.

a. How many x-intercepts does $y = x^2 + 2x - 5$ have?

b. Approximately where does the graph of $y = x^2 + 2x - 5$ cross the x-axis?

9-23. This is a checkpoint for finding the equation for the inverse of a function.

Consider the function $f(x) = 2\sqrt{3(x-1)} + 5$.

a. Find the equation for the inverse of $f(x)$.

b. Sketch the graph of both the original and the inverse.

c. Check your results by referring to the Checkpoint 17 materials located at the back of your book.

If you needed help to write the equation of the inverse of this function, then you need more practice with inverses. Review the Checkpoint 17 materials and try the practice problems. Also, consider getting help outside of class time. From this point on, you will be expected to write equations for inverses of functions such as this one quickly and easily.

9-24. Graph the inequality $x^2 + y^2 \le 25$, and then describe its graph in words.

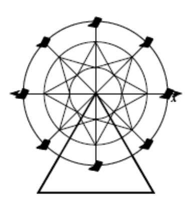
9-25. Find x if $2^{p(x)} = 4$ and $p(x) = x^2 - 4x - 3$.

9-26. Start with the graph of $y = 3^x$, then write new equations that will shift the graph as described below.

a. Down 4 units. b. Right 7 units.

9-27. THE COUNTY FAIR FERRIS WHEEL

Consider this picture of a Ferris wheel. The wheel has a 60-foot diameter and is drawn on a set of axes with the Ferris wheel's hub (center) at the origin. Use a table like the one below and draw a graph that relates the angle (in standard position) of the spoke leading to your seat to the approximate height of the top of your seat above or below the height of the central hub. The table below starts at –90°, your starting position before you ride around the wheel.



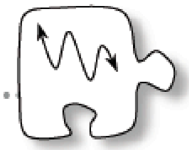
x (angle)	–90°	–45°	0°	45°	90°	135°	180°	....	270°
y (height)	–30'								

a. The wheel goes around (counter-clockwise) several times during a ride. How could you reflect this fact in your graph? Update your graph.

b. What is the maximum distance above or below the center that the top of your seat attains during the ride?

c. Find an equation to fit the County Fair Ferris wheel ride.

9.1.2 How can I predict the graph?



More Graphs of Polynomials

Today you will use what you learned in the "Polynomial Function Investigation" to respond to some questions. Thinking about how to answer these questions should help you clarify and expand on some of your ideas as well as help you learn how to use the vocabulary involved with polynomials.

9-28. Use your finger to trace an approximate graph of polynomial functions in the air, as directed by your teacher. Or you may sketch each of the polynomial functions below quickly on paper. Just sketch the graph without the x- and y- axes.

a. $P(x)=(x+10)(x+7)(x-12)$

b. $Q(x)=(x+6)(x+3)(x-5)(x-8)$

c. $R(x)=-(x+4)(x+2)(x-6)(x-10)$

d. $W(x)=(x+7)^2(x-7)^2$

e. $S(x)=(x+6)(x+3)(x-5)(x-8)(x-12)$

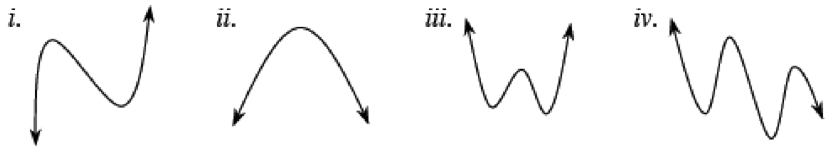
9-29. Look back at the work you did in Lesson 9.1.1 (problem 9-2, "Polynomial Function Investigation"). Then answer the following questions.

a. What is the maximum number of roots a polynomial of degree 3 can have? Sketch an example.

b. What do you think is the maximum number of roots a polynomial of degree n can have?

c. Can a polynomial of degree n have fewer than n roots? Under what conditions?

9-30. For each polynomial function shown below, state the minimum degree its equation could have.



a. Which of the graphs above show that as the x-values get very large the y-values continue to get larger and larger?

b. How would you describe the other graphs for very large x-values?

c. When the y-values of a graph get very large as the x-values get large, the graph has **positive orientation**. When the y-values of a graph get very small as the x-values get large, the graph has **negative orientation**. How is each of the above graphs oriented?

9-31. For each graph in problem 9-30, you decided what the minimum degree of its equation could be. Under what circumstances could graphs that look the same as these have polynomial equations of a *higher* degree?

Consider the graphs of $y=(x-1)^2$ and $y=(x-1)^4$.

a. How are these graphs similar? How are they different?

b. Could the equation for graph (*ii*) from the previous problem have degree 4?

c. Could it have degree 5? Explain.

d. How is the graph of $y=x^3$ similar to or different from the graph of $y=x^5$?

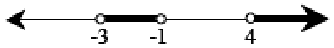
e. How do the shapes of graphs of $y=(x-2)^3$ and $y=(x+1)^5$ differ from the shapes of graphs of equations that have three or five factors that are all different?

9-32. In $P_1(x) = (x-2)(x+5)^2$ (the first example from the "Polynomial Function Investigation"), $(x+5)^2$ is a factor. This produces what is called a **double root** of the function.

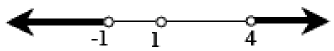
a. What effect does this have on the graph?

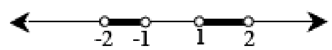
b. Check your equations for a **triple root**. What effect does a triple root have on the graph?

9-33. We can use a number line to represent the x-values for which a polynomial graph is above or below the x-axis. The bold parts of each number line below show where the output values of a polynomial function are positive (that is, where the graph is above the x-axis). The open circles show locations of the x-intercepts or roots of the function. Where there is no shading, the value of the function is negative. Sketch a possible graph to fit each number line, and then write a possible equation. (Each number line represents the x-axis for a different polynomial.)

a. 

b. 

c. 

d. 

9-34. What can you say about the graphs of polynomial functions with an even degree compared to the graphs of polynomial functions with an odd degree? Use graphs from the "Polynomial Functions Investigation" (and maybe some others), to **justify** your response.

9-35. Choose three of the polynomials you graphed in the "Polynomial Functions Investigation" (problem 9-2) and create number lines for their graphs similar to the ones in problem 9-33.

9-36. Create a new number-line description (like the ones in problem 9-33) and then trade with a partner. (Each team member should create a different number line.) After you have traded, find a possible graph and equation for a polynomial function to fit the description you have received. Then **justify** your results to your team and check your team members' results.

9-37. Without using a calculator, sketch rough graphs of the following functions.

a. $P(x) = -x(x+1)(x-3)$

b. $P(x) = (x-1)^2(x+2)(x-4)$

c. $P(x) = (x+2)^3(x-4)$

METHODS AND MEANINGS

MATH NOTES

Roots and Zeros

The **roots** of a polynomial function, $p(x)$, are the **solutions** of the equation $p(x)=0$. Another name for the roots of a function is **zeros of the function** because at each root, the value of the function is zero. The real roots (or zeros) of a function have the same value as the x-values of the x-intercepts of its graph because the x-intercepts are the points where the y-value of the function is zero.

Sometimes roots can be found by factoring. In the "Parabola Lab" (problems 4-13 and 4-14), you discovered how to make a parabola "sit" on the x-axis (the polynomial has one root), and you looked at ways of making parabolas intersect the x-axis in two specific places (two roots).

Review & Preview

9-38. Where does the graph $y=(x+3)^2-5$ cross the x-axis?

9-39. If you were to graph the function $f(x)=(x-74)^2(x+29)$, where would the graph intersect the x-axis?

9-40. For each pair of intercepts given below, write an equation for a quadratic function in standard form.

a. $(-3,0)$ and $(2,0)$

b. $(-3,0)$ and $(\frac{1}{2},0)$

9-41. What is the degree of each polynomial function below?

a. $P(x)=0.08x^2+28x$

b. $y=8x^2-\frac{1}{7}x^5+9$

c. $f(x)=5(x+3)(x-2)(x+7)$

d. $y=(x-3)^2(x+1)(x^3+1)$

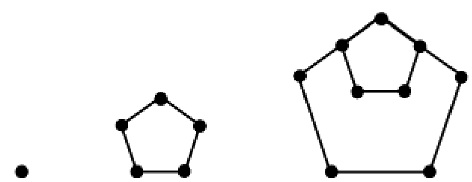
9-42. Are parabolas polynomial functions? Are lines polynomial functions? Are cubics? Exponentials? Circles? In all cases, explain why or why not.

9-43. A sequence of pentagonal numbers is started at right.

a. Find the next three pentagonal numbers.

b. What kind of sequence do the pentagonal numbers form?

c. What is the equation for the n^{th} pentagonal number?



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9-44. A circle with its center on the line $y=3x$ in the 1^{st} quadrant is tangent to the y-axis.

a. If the radius is 2, what is the equation of the circle?

b. If the radius is 3, what is the equation of the circle?

9-45. Sketch the graph of each function below on the same set of axes.

a. $y=2^x$

b. $y=2^x+5$

c. $y=2^x-5$

9-46. For each equation, find two solutions $0^\circ \le \theta < 360^\circ$, which make the equation true. You should not need a calculator.

a. $\sin\theta=\frac{1}{2}$

b. $\tan\theta=\sqrt{3}$

c. $\cos\theta=\frac{\sqrt{3}}{2}$

d. $\sin\theta=-\frac{\sqrt{2}}{2}$

METHODS AND MEANINGS

MATH NOTES

Polynomials, Degree, Coefficients

A **polynomial** in one variable is an expression that can be written as the sum or difference of terms of the form:

$$(\text{any number}) \cdot x^{(\text{whole number})}$$

Polynomials with one variable (often x) are usually arranged with powers of x in order, starting with the highest, left to right. Polynomials can include only the operations of addition, subtraction, or multiplication.

The highest power of the variable in a polynomial of one variable is called the **degree** of the polynomial. The numbers that multiply each term are called **coefficients**. See the examples below.

Example 1: $f(x) = 7x^5 + 2.5x^3 - \frac{1}{2}x + 7$ is a polynomial function of degree 5 with coefficients 7, 0, 2.5, 0, $-\frac{1}{2}$, and 7. Note that the last term, 7, is called the **constant** term but represents the variable expression $7x^0$, since $x^0 = 1$.

Example 2: $y = 2(x+2)(x+5)$ is a polynomial in factored form with degree 2 because it can be written in standard form as $y = 2x^2 + 14x + 20$. It has coefficients 2, 14, and 20.

The following are not polynomial functions: $y = 2^x - 3$, $f(x) = \frac{1}{x^2 - 2x} + x$, and $y = \sqrt{x-2}$.

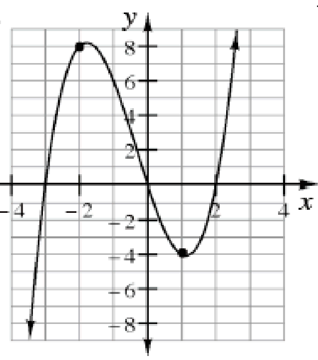
9.1.3 How can I find the equation?

Stretch Coefficients for Polynomial Functions

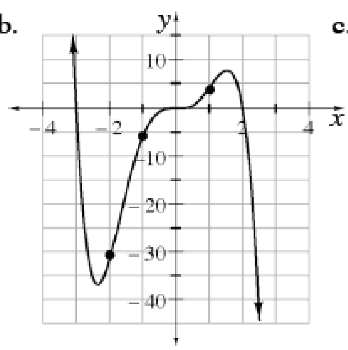
In Lesson 9.1.2 you found possible equations for the graphs of polynomial functions based on their x-intercepts. Many of the sketches you used did not even include the scale on the y-axis. In this lesson, you will focus on figuring out equations that represent *all* of the points on the graphs.

9-47. Find reasonable equations for each of the following polynomial functions. Without using a graphing calculator, how can you check the accuracy of your equations? How can you check to see whether the y-values (or stretch factor) are accurate? Show how you checked the accuracy in each case. Were your equations accurate?

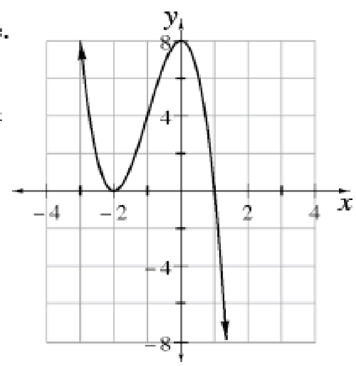
a.



b.



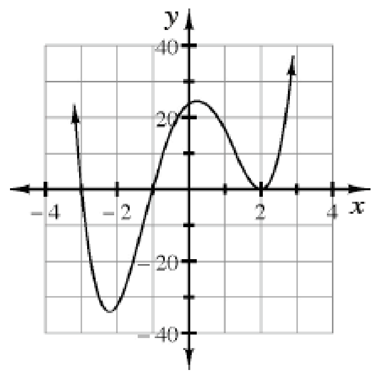
c.



9-48. What is the difference between the graphs of the functions $y = x^2(x-3)(x+1)$ and $y = 3x^2(x-3)(x+1)$?

9-49. ARE THE INTERCEPTS ENOUGH?

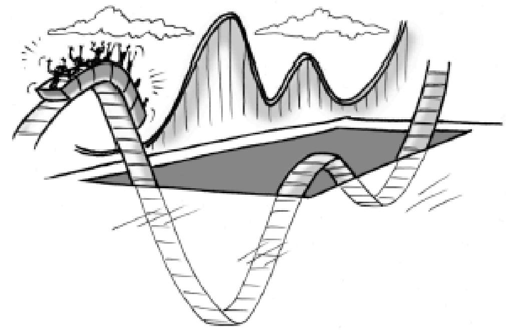
Melvin wrote the equation $y=(x+3)(x+1)(x-2)^2$ to represent the graph at right. How well does this equation represent the graph?



a. Explain how you can decide how well the equation represents the graph. What can you do to the equation to make it a better fit for the graph? What equation would fit better?

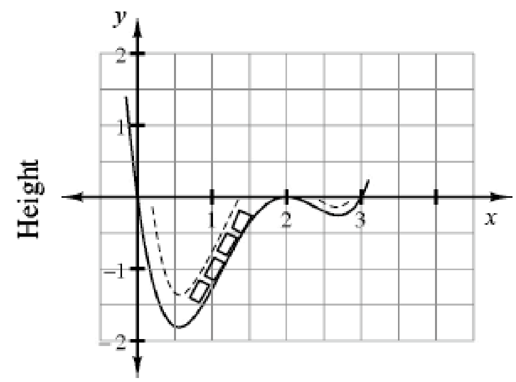
b. Before you figured it out, you could have written the polynomial for this graph as $P(x)=a(x+3)(x+1)(x-2)^2$. What if you did not have a graphing calculator, but you were told that the graph goes through the point (1, 16)? How could you use that information to determine the exact equation? Once you have decided on a method with your team, try it. How can you test the accuracy of your equation?

9-50. THE COUNTY FAIR COASTER RIDE



Now that you have more expertise with polynomial equations and their graphs, the Mathamericaland Carnival Company has hired your team to find the *exact* equation to represent its roller-coaster track.

The numbers along the x-axis are in hundreds of feet. At 250 feet, the track will be 20 feet below the surface. This gives the point $(2.5, -0.2)$.



a. What degree polynomial represents the portion of the roller coaster represented by the graph at right?

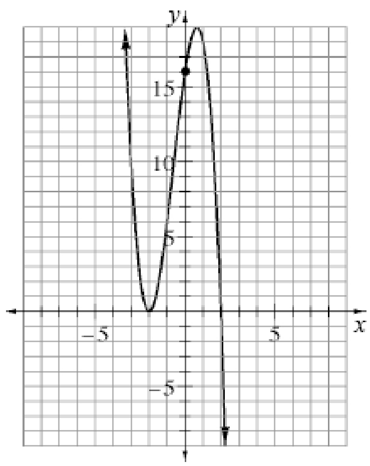
b. What are the roots?

c. Find an exact equation for the polynomial that will generate the curve of the track.

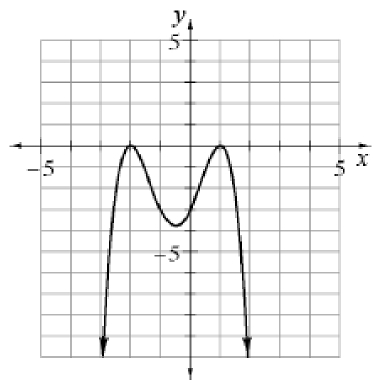
d. What is the deepest point of the roller coaster's tunnel?

9-51. Write an exact equation for each graph below.

a.



b.



9-52. Write a polynomial equation for a function with a graph that bounces off the x-axis at $(-1, 0)$, crosses it at $(4, 0)$, and goes through the point $(-2, -18)$.

9-53. Armando came up with the equation $y = 3(x+1)^4(x-4)$ for problem 9-52. Does his equation fit all of the given criteria? Why or why not? Is it the same as the equation you came up with?

9-54. What if problem 9-52 also had said that the graph went through the point $(1, -36)$? Is there still more than one possible equation? Explain.

9-55. What information about the graph of a polynomial function is necessary to determine exactly one correct equation? Discuss this with your team.

MATH NOTES

LOOKING DEEPER

Notation for Polynomials

A general equation to represent all polynomials: The general equation of a second-degree (quadratic) polynomial is often written in the form $f(x)=ax^2+bx+c$, and the general equation of a third-degree (cubic) polynomial is often written in the form $f(x)=ax^3+bx^2+cx+d$.

For a polynomial with an undetermined degree n, it is unknown how many letters will be needed for the coefficients, so instead of using a, b, c, d, e, etc., mathematicians use only the letter a, and they put on subscripts, as shown below.

$$f(x)=(a_n)x^n+(a_{n-1})x^{(n-1)}+\ldots+(a_1)x^1+a_0$$

This general polynomial has degree n and coefficients $a_n,\ a_{n-1},\ \ldots,\ a_1,\ a_0$.

For example, for $7x^4-5x^3+3x^2+7x+8$, the degree is 4. In this specific case, a_n is a_4 and $a_4=7$, a_{n-1} is $a_3=-5$, a_{n-2} is $a_2=3$, $a_1=7$, and $a_0=8$.

May 01, 2011

Review & Preview

9-56. What is the stretch factor (the value of a) for the equation of the graph in part (c) of problem 9-47? Write the exact equation of the function.

9-57. For each of the following polynomial expressions, find the degree, list the coefficients, and then label them a_0 through a_n. Refer to the example in the Lesson 9.1.3 Math Notes box above about polynomial notation.

a. $6x^4 - 3x^3 + 5x^2 + x + 8$

b. $-5x^3 + 10x^2 + 8$

c. $-x^2 + x$

d. $x(x-3)(x-5)$

e. x

f. 10

9-58. Write a polynomial equation for a graph that has three x-intercepts: $(-4, 0)$, $(1, 0)$, and $(3, 0)$, and passes through the point $(-1, 60)$.

9-59. The x-intercepts of a quadratic polynomial are given below. Find a possible quadratic equation in standard form.

a. $x = \frac{3}{4}$, $x = -2$

b. $x = -\sqrt{5}$, $x = \sqrt{5}$

9-60. Consider the functions $y = \frac{1}{2}$ and $y = \frac{16}{x^2-4}$. Find the coordinates where the graphs of the functions intersect.

9-61. Find the center and radius of each circle below.

a. $(y-7)^2 = 25 - (x-3)^2$

b. $x^2 + y^2 + 10y = -9$

c. $x^2 + y^2 + 18x - 8y + 47 = 0$

d. $y^2 + (x-3)^2 = 1$

9-62. Without using a calculator, write the solution to each equation.

a. $2^x = 17$

b. $\log_3(x+1) = 5$

c. $\log_3(3^x) = 4$

d. $4^{\log_4(x)} = 7$

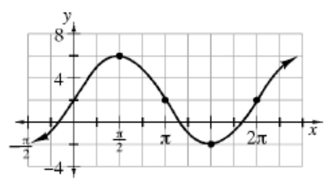
9-63. Consider the function $y = x^2 + 5x + 7$.

a. Complete the square to find the vertex.

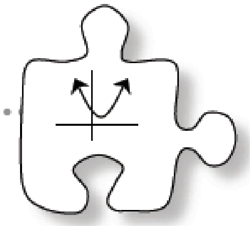
b. Find the y-intercept.

c. Use the vertex, the y-intercept, and the symmetry of parabolas to find a third point and sketch the graph.

9-64. Write a possible equation for the graph at right.



9.2.1 What are imaginary numbers?



Introducing Imaginary Numbers

In the past, you have not been able to solve some quadratic equations like $x^2+4=0$ and $x^2+1=0$, because there are no real numbers you can square to get a negative answer. To solve this issue, mathematicians created a new, expanded number system based on one new number. But this was not the first time mathematicians had invented new numbers! To read about other such inventions, refer to the Math Notes box that follows problem 9-65.

In this lesson, you will learn about imaginary numbers and how you can use them to solve equations you were previously unable to solve.

9-65. Consider the equation $x^2=2$.

a. How do you "undo" squaring a number?

b. When you solve $x^2=2$, how many solutions should you get?

c. How many x-intercepts does the graph of $y=x^2-2$ have?

d. Solve the equation $x^2=2$. Write your solutions both as radicals and as decimal approximations.

MATH NOTES

LOOKING DEEPER

Historical Note: Irrational Numbers

In Ancient Greece, people believed that all numbers could be written as fractions of whole numbers (what are now called **rational numbers**). Many individuals realized later that some numbers could not be written as fractions (such as $\sqrt{2}$), and these individuals challenged the accepted beliefs. Some of the people who challenged the beliefs were exiled or outright killed over these challenges.

The Greeks knew that for a one-unit square, the length of the diagonal, squared, yielded 2. When it was shown that no rational number could do that, the existence of what are called *irrational numbers* was accepted and symbols like $\sqrt{2}$ were invented to represent them.

The problem $x^2 = 3$ also has no rational solutions; fractions can never work exactly. The rational (i.e., decimal) solutions that calculators and computers provide are only approximations; the exact answer can only be represented in radical form, namely, $\pm\sqrt{3}$.

9-66. Mathematicians throughout history have resisted the idea that some equations may not be solvable. Still, it makes sense that $x^2+1=0$ cannot be solved because the graph of $y=x^2+1$ has no x-intercepts. What happens when you try to solve $x^2+1=0$?

MATH NOTES

LOOKING DEEPER

Historical Note: Imaginary Numbers

In some ways, each person's math education parallels the history of mathematical discovery. When you were much younger, if you were asked, *"How many times does 3 go into 8?"* or *"What is 8 divided by 3?"* you might have said, *"3 doesn't go into 8."* Then you learned about numbers other than whole numbers, and the question had an answer. Of course, in some situations you are only interested in whole numbers, and then the first answer is still the right one. Later, if you were asked, *"What number squared makes 5?"* you might have said, *"No number squared makes 5."* Then you learned about numbers other than rational numbers, and you could answer that question.

Similarly, until about 500 years ago, the answer to the question, *"What number squared makes –1?"* was, *"No number squared makes –1."* Then something remarkable happened. An Italian mathematician named Bombelli used a formula for finding the roots of third-degree polynomials. Within the formula was a square root, and when he applied the formula to a particular equation, the number under the square root came out negative. Instead of giving up, he had a brilliant idea. He had already figured out that the equation had a solution, so he decided to see what would happen if he pretended that there *was* a number he could square to make a negative. Remarkably, he was able to continue the calculation, and eventually the "imaginary" number disappeared from the solution. More importantly, the resulting answer worked; it solved his original equation. This led to the acceptance of these so-called **imaginary numbers**. The name stuck, and mathematicians became convinced that all quadratic equations do have solutions. Of course, in some situations you will only be interested in real numbers (that is, numbers not having an imaginary part), and then the original answer, that there is no solution, is still the correct one.

9-67. In the 1500s, an Italian mathematician named Rafael Bombelli invented the imaginary number $\sqrt{-1}$, which is now called i. $\sqrt{-1} = i$ implies that $i^2 = -1$. After this invention, it became possible to find solutions for $x^2 + 1 = 0$; they are i and $-i$. What would be the value of $\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16i^2} = ?$ Use the definition of i to rewrite each of the following expressions.

a. $\sqrt{-4}$ b. $(2i)(3i)$ c. $(2i)^2(-5i)$ d. $\sqrt{-25}$

9-68. Graph the function $y = x^2 - 4x + 5$.

a. Does the graph cross the x-axis? Should the equation $x^2 - 4x + 5 = 0$ have real solutions?

b. Use the Quadratic Formula to solve $x^2 - 4x + 5 = 0$. Use your new understanding of imaginary numbers to simplify your results as much as possible.

c. A real number plus (or minus) a multiple of i (like each of the solutions to $x^2 - 4x + 5 = 0$) is called a **complex number**. Check one of your solutions from part (b) by substituting it into the equation for x and simplifying the result.

9-69. When a graph crosses the x-axis, the x-intercepts are often referred to as the **real roots** of the equation that results when $y = 0$. You have seen that solutions to equations can be real or complex, so it follows that roots can also be real or complex. Compare and contrast what happens with the graphs and equations for the three cases in parts (a) through (c) below.

a. Sketch the graph of $y = (x + 3)^2 - 4$. What are the roots?

b. Sketch the graph of $y = (x + 3)^2$. What are the roots?

c. Sketch the graph of $y = (x + 3)^2 + 4$. Can you find the roots by looking at the graph? Why or why not? Find the roots by solving $(x + 3)^2 + 4 = 0$.

d. Make **general** statements about the relationship between graphs of parabolas and the kinds of roots their equations have.

9-70. Consider the equations $y = x^2$ and $y = 2x - 5$.

a. On one set of axes, sketch the graphs and label the intersection.

b. Use algebra to solve the system of equations.

c. Discuss your results with your team. What could these solutions mean?

9-71. Do the graphs of $y = \frac{1}{x}$ and $y = -x + 1$ intersect? What kind of algebraic solutions will this system have? Verify your answer by solving the system.

MATH NOTES

METHODS AND MEANINGS

Imaginary and Complex Numbers

The **imaginary number** that solves the equation $x^2 = -1$ is i, so $i^2 = -1$, and the two solutions of the equation are i and $-i$.

In general, i follows the rules of real number arithmetic. The sum of two imaginary numbers is imaginary (unless it is 0). Multiplying the imaginary number i by every possible real number would yield the set of all the imaginary numbers.

The set of numbers that solve equations of the form $x^2 =$ (a negative real number) is called the set of **imaginary numbers**. Imaginary numbers are not positive, negative, or zero. The collection (set) of positive and negative numbers (integers, rational numbers (fractions), and irrational numbers), are referred to as the **real numbers**.

The sum of a real number (other than zero) and an imaginary number, such as $2+i$, is generally neither real nor imaginary. Numbers such as these, which can be written in the form $a+bi$, where a and b are real numbers, are called **complex numbers**. Each complex number has a real component, a, and an imaginary component, bi. The real numbers are considered to be complex numbers with $b=0$, and the imaginary numbers are complex numbers with $a=0$.

Review & Preview

9-72. Write each of the following expressions in the form $a+bi$.

a. $-18-\sqrt{-25}$

b. $\frac{2\pm\sqrt{-16}}{2}$

c. $5+\sqrt{-6}$

9-73. Explain why $i^3=-i$. What does i^4 equal?

9-74. If $f(x)=x^2+7x-9$, calculate the values in parts (a) through (c) below.

a. $f(-3)$

b. $f(i)$

c. $f(-3+i)$

9-75. Is $5+2i$ a solution to $x^2-10x=-29$? How can you be sure?

9-76. Solve $16^{(x+2)}=8^x$.

9-77. Is $(x-5)^2$ equivalent to $(5-x)^2$? Explain why or why not.

9-78. Calculate the value of each expression below.

a. $\sqrt{-49}$

b. $\sqrt{-2}$

c. $(4i)^2$

d. $(3i)^3$

9-79. Find the inverse functions below.

a. If $f(x)=2x-3$, then what does $f^{-1}(x)$ equal?

b. If $h(x)=(x-3)^2+2$, then what does $h^{-1}(x)$ equal?

9-80. Solve each equation.

a. $5.2(3.75)^x=100$

b. $4+3x^4=81$

9.2.2 What are complex roots?

Complex Roots

In this lesson, you will solve equations as well as **reverse your thinking** to **investigate** the relationship between the complex solutions to a quadratic equation and the equation they came from.

9-81. Find the roots of each of the following quadratic functions by solving for x when $y = 0$. Does the graph of either of these functions intersect the x-axis?

a. $y = (x + 5)^2 + 9$

b. $y = x^2 - 4x + 9$

9-82. What do you notice about the complex solutions in problem 9-81? Describe any patterns you see. Discuss these with your team and write down everything you can think of.

9-83. In parts (a) through (d) below, look for patterns as you calculate the sum and the product for each pair of complex numbers. Use what you find to answer parts (e) through (g).

a. $2 + i,\ 2 - i$

b. $3 - 5i,\ 3 + 5i$

c. $-4 + i,\ -4 - i$

d. $1 + i\sqrt{3}, 1 - i\sqrt{3}$

e. What complex number can you multiply $3 + 2i$ by to get a real number?

f. What happens when you multiply $(-4 + 5i)(-4 + 3i)$?

g. What complex number can you multiply $a + bi$ by to get a real number?

9-84. WHAT EQUATION HAS THESE SOLUTIONS?

Each of the four pairs of complex numbers in problem 9-83 could be the roots of a quadratic function.

Your task: With your team, create a quadratic equation for each pair of complex numbers in parts (a) through (d) of problem 9-83 such that those numbers are the roots. Discuss the methods you use for writing the equations and write summary statements describing your methods.

Discussion Points

How can we **reverse** the process of solving and work backwards?

How can we use what we know about factors and zeros?

How are the solutions related to the standard form of the equation?

Further Guidance

9-85. Problem 9-83 made Mariposa curious about sums and products. She decided to solve the equation $x^2 - 6x + 25 = 0$ and look at the sums and products of its solutions. What patterns can you help her find that might give her ideas about the equation once she knows the solutions?

9-86. Austin had another idea. He knew that if 3 and –5 were solutions of a quadratic equation then $(x-3)$ and $(x+5)$ would be factors that could be multiplied to get a quadratic polynomial. How could his idea be used with the pairs of complex solutions in problem 9-83? Choose one pair and show how to use your idea.

9-87. Melvin had still another idea. *"Why not just let $x = -4 \pm i$ and work backwards?"* He asked. Would his idea work?

Further Guidance section ends here.

9-88. For each pair of numbers below, find a quadratic equation that has these numbers as solutions.

a. $\frac{3}{4}$ and -5

b. $3i$ and $-3i$

c. $5+2i$ and $5-2i$

d. $-3+\sqrt{2}$ and $-3-\sqrt{2}$

METHODS AND MEANINGS

MATH NOTES

The Discriminant and Complex Conjugates

For any quadratic equation $ax^2 + bx + c = 0$, you can determine whether the roots are real or complex by examining the part of the quadratic formula that is under the square-root sign. The value of $b^2 - 4ac$ is known as the **discriminant**. The roots are real when $b^2 - 4ac \geq 0$ and complex when $b^2 - 4ac < 0$.

For example, in the equation $2x^2 - 3x + 5 = 0$, $b^2 - 4ac = (-3)^2 - 4(2)(5) = -31 < 0$, so the equation has two complex roots and the parabola $y = 2x^2 - 3x + 5$ does not intersect the x-axis.

Complex roots of quadratic equations with real coefficients will have the form $a - bi$ and $a + bi$, which are called **complex conjugates**. The sum and product of two complex conjugates are always real numbers.

For example, the conjugate for the complex number $-5 + 3i$, is $-5 - 3i$. $(-5 + 3i) + (-5 - 3i) = -10$ and $(-5 + 3i)(-5 - 3i) = 25 - 9i^2 = 34$.

Note: With the introduction of complex numbers, the use of the terms *roots* and *zeros* of polynomials expands to include complex numbers that are solutions of the equations when $p(x) = 0$.

Review & Preview

9-89. For each of the following sets of numbers, find the equation of a function that has these numbers as roots.

a. $-3+i$ and $-3-i$

b. $5+\sqrt{3}$ and $5-\sqrt{3}$

c. -2, $\sqrt{7}$, and $-\sqrt{7}$

d. 4, $-3+i$, and $-3-i$

9-90. Raul claims that he has a shortcut for deciding what kind of roots a function has. Jolene thinks that a shortcut is not possible. She says you just have to solve the quadratic equation to find out. They are working on $y=x^2-5x-14$.

Jolene says, *"See, I just start out by trying to factor. This one can be factored $(x-7)(x+2)=0$, so the equation will have two real solutions and the function will have two real roots."*

"But what if it can't be factored?" Raul asked. *"What about $x^2+2x+2=0$?"*

"That's easy! I just use the Quadratic Formula," says Jolene. *"And I get... let's see... negative two plus or minus the square root of... two squared... that's 4... minus... eight..."*

"Wait!" Raul interrupted. *"Right there, see, you don't have to finish. 2^2 minus $4\cdot 2$, that gives you -4. That's all you need to know. You'll be taking the square root of a negative number so you will get a complex result."*

"Oh, I see," said Jolene. *"I only have to do part of the solution, the part you have to take the square root of."*

Use Raul's method to tell whether each of the following functions has real or complex roots without completely solving the equation. Note: Raul's method is summarized in the Math Notes box for this lesson.

a. $y=2x^2+5x+4$

b. $y=2x^2+5x-3$

9-91. Decide which of the following equations have real roots, and which have complex roots without completely solving them.

a. $y=x^2-6$

b. $y=x^2+6$

c. $y=x^2-2x+10$

d. $y=x^2-2x-10$

e. $y=(x-3)^2-4$

f. $y=(x-3)^2+4$

9-92. Consider this geometric sequence: $i^0, i^1, i^2, i^3, i^4, i^5, \ldots, i^{15}$.

a. You know that $i^0=1$, $i^1=i$, and $i^2=-1$. Calculate the result for each term up to i^{15}, and describe the pattern.

b. Use the pattern you found in part (a) to calculate i^{16}, i^{25}, i^{39}, and i^{100}.

c. What is i^{4n}, where n is a positive whole number?

d. Based on your answer to part (c), simplify i^{4n+1}, i^{4n+2}, and i^{4n+3}.

e. Calculate i^{396}, i^{397}, i^{398}, and i^{399}.

9-93. Use the pattern from the previous problem to help you to evaluate the following expressions.

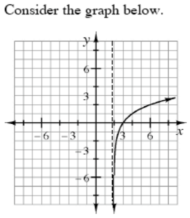
a. i^{592}

b. i^{797}

c. $i^{10,648,202}$

9-94. Describe how you would evaluate i^n where n could be any integer.

9-95. Consider the graph below.

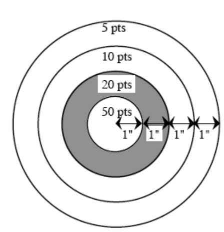


a. What is the parent for this function?

b. What is the equation of the vertical asymptote?

c. Write a possible equation for this graph.

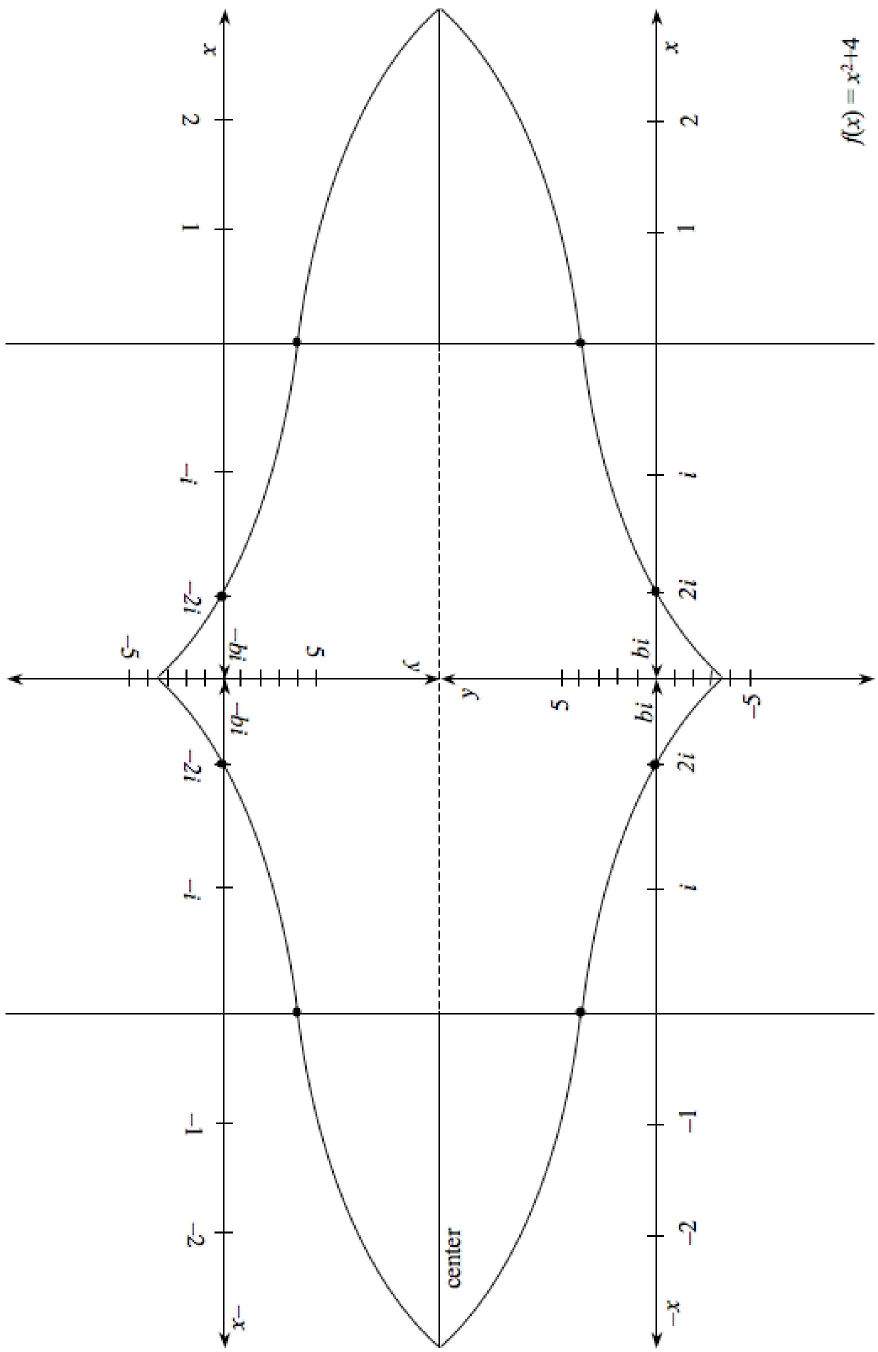
9-96. In one of the games at the county fair, people pay to shoot a paint pistol at the target shown at right. The center has a radius of one inch. Each concentric circle has a radius one inch larger than the preceding circle. Assuming the paint pellet hits the target randomly, what is the probability that it hits:



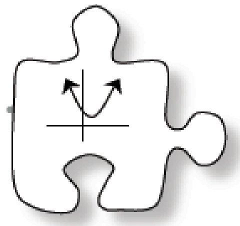
a. The 50-point ring?

b. The 20-point ring?

9-97. Given $C=\begin{bmatrix}3 & -2\\ -5 & 1\end{bmatrix}$, $D=\begin{bmatrix}0 & 3\\ 4 & -3\end{bmatrix}$ find:


f(x) = x²+4
center
x
−x
y
bi
−bi
2i
−2i
i
−i
5
−5
1
2
−1
−2

9.2.3 Where are the complex numbers?



More Complex Numbers and Equations

If the number line is filled with real numbers, how can imaginary and complex numbers be represented geometrically? In this lesson, you will learn a way to graph complex numbers and interpret the graphical meaning of complex solutions.

9-98. Avi and Tran were trying to figure out how they could represent complex numbers geometrically. Avi decided to make a number line horizontal like the x-axis to represent the real part as well as a vertical line like the y-axis to represent the imaginary part.

a. Draw a set of axes and label them as Avi described.

b. How could Avi and Tran graph a point to represent the complex number $3+4i$? Be prepared to share your **strategies** with the class.

c. Use the method you developed in part (b) to plot points to represent the six numbers below.

i.	$2+5i$	*ii.*	$6-i$	*iii.*	$-5-3i$
iv.	4	*v.*	$7i$	*vi.*	$-4+2i$

9-99. On a new set of **complex axes** (like those drawn by Avi and Tran in problem 9-98), locate points representing all of the following complex numbers

a. $3+4i$, $3-4i$, $-3+4i$, and $-3-4i$.

b. The four complex numbers represented by $\pm 4 \pm 3i$.

c. $5,\ -5,\ 5i,$ and $-5i$

d. What do you notice about your graph? How far from (0, 0) is each point?

9-100. On the real number line, the distance from 0 to a point on the line is defined as the absolute value of the number. Similarly, in the **complex plane** (the plane defined by a set of complex axes), the **absolute value** of a complex number is its distance from zero or the origin (0, 0). In the previous problem, the absolute value of all of those complex numbers was 5. For each of the following questions, a sketch in the complex plane will help in visualizing the result.

a. What is the absolute value of $-8+6i$?

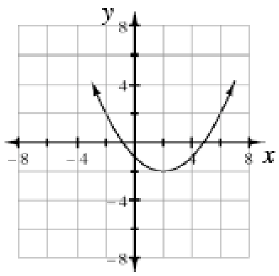
b. What is the absolute value of $7-2i$?

c. What is $|4+i|$?

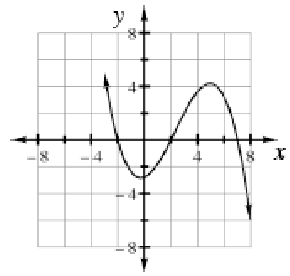
d. What is the absolute value of $a+bi$?

9-101. Based on the following graphs, how many *real* roots does each polynomial function have?

a.

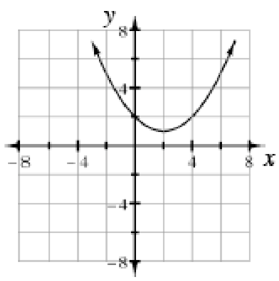


b.

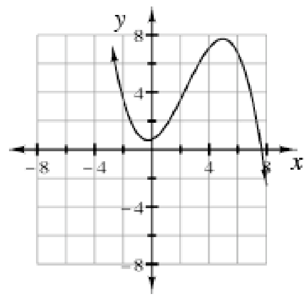


Graphs (a) and (b) above have been vertically shifted to create graphs (c) and (d) shown below. How many *real* roots does each of these new polynomial functions have?

c.

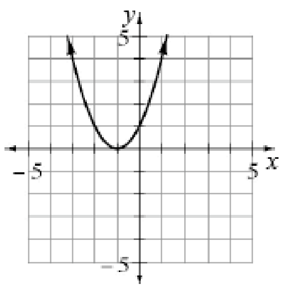


d.

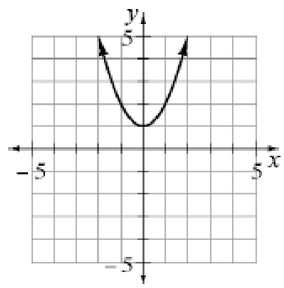


The polynomials in parts (c) and (d) do not have fewer roots. Polynomial (c) still has *two* roots, but now the roots are complex. Polynomial (d) has *three* roots: two are complex, and only one is real.

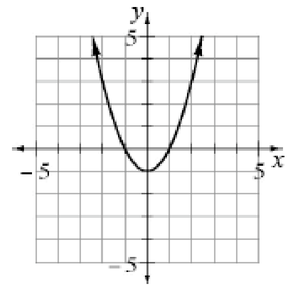
9-102. Recall that a polynomial function with degree n crosses the x-axis at most n times. For instance, $y=(x+1)^2$ intersects the x-axis once, while $y=x^2+1$ does not intersect it at all. The function $y=x^2-1$ intersects it twice. These graphs are shown below.



$y=(x+1)^2$



$y=x^2+1$



$y=x^2-1$

a. A third-degree equation might intersect the x-axis one, two, or three times. Make sketches of all these possibilities.

b. Can a third-degree equation have zero real roots? Explain why or why not.

9-103. Now consider the graph of $y = x^3 - 3x^2 + 3x - 2$.

a. How many real solutions could $x^3 - 3x^2 + 3x - 2 = 0$ have?

b. Check to verify that $x^3 - 3x^2 + 3x - 2 = (x-2)(x^2 - x + 1)$.

c. Find all of the solutions of $x^3 - 3x^2 + 3x - 2 = 0$.

d. How many x-intercepts does $y = x^3 - 3x^2 + 3x - 2$ have? How many real roots and how many non-real roots (complex)?

9-104. Sketch the graph of $f(x) = x^2 + 4$ and solve the equation $x^2 + 4 = 0$ to find its roots.

a. Describe the parabola. Be sure to include the vertex and the equation of its axis of symmetry.

b. With a partner, obtain a copy of the Lesson 9.2.3 Resource Page from your teacher, and follow the directions below to make a 3-D model that will show the location of the complex roots in a complex plane that is perpendicular to the real plane in which you drew the graph of the parabola.

i. Fold the paper on the line marked ***bi*** and –***bi.*** This is a "mountain" fold, so the printing is on the outside.

Problem continues on next page.→

9-104. *Problem continues from previous page.*

ii. Cut the paper exactly along the dotted line. Do not cut beyond the dotted portion.

iii. Now make "valley" folds on the two lines parallel to the first fold.

iv. Hold the two ends of the paper and push them toward the center so the center pops up, and then fold the top and bottom of the paper back on the line marked "center."

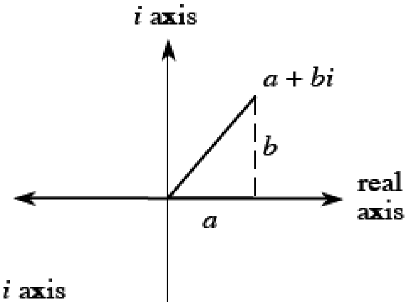
You should have a three-dimensional coordinate system with the *xy*-plane facing you and the *i*-axis coming out toward you. The equation $f(x) = x^2 + 4$ should be in the lower right corner. Now locate the roots of the function on your 3-D model.

MATH NOTES

METHODS AND MEANINGS

Graphing Complex Numbers

To represent imaginary numbers, an imaginary axis and a plane are needed. Real numbers are on the horizontal axis and imaginary numbers are on the vertical axis, as shown in the examples below. Complex numbers are graphed using the same method that coordinate points are: the number $2+3i$ is located at (2, 3), and the number i or $0+1i$ is located at $(0, 1)$. The number -2 or $-2+0i$ is located at $(-2,0)$. This representation is called the **complex plane**.



In the complex plane, $a+bi$ is located at the point (a, b). Its distance from the origin is its **absolute value.** To find the absolute value, calculate the distance from (0, 0) to (a, b):

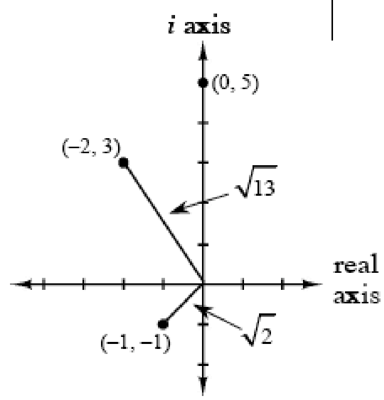
$$|a+bi|=\sqrt{a^2+b^2}$$

Examples:

$$|-2+3i|=\sqrt{(-2)^2+3^2}=\sqrt{13}$$

$$|-1-i|=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$$

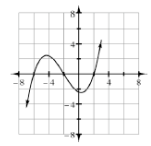
$$|5i|=5$$



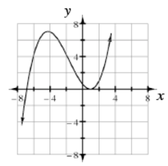
Review & Preview

9-105. In parts (a) through (d) below, for each polynomial function $f(x)$, the graph of $f(x)$ is shown. Based on this information, tell how many linear and quadratic factors the factored form of its equation should have and how many real and complex (non-real) solutions $f(x)=0$ might have. (Assume a polynomial function of the lowest possible degree for each one.)

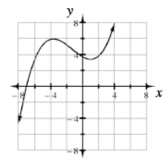
Example: $f(x)$ at right will have three linear factors, therefore three real roots and no complex roots.



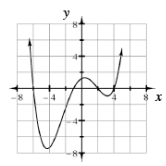
a.



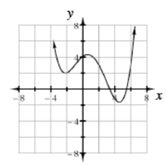
b.



c.



d.



9-106. Make a sketch of a graph $p(x)$ so that $p(x)=0$ would have the indicated number and type of solutions.

a. 5 real solutions

b. 3 real and 2 complex

c. 4 complex

d. 4 complex and 2 real

e. For parts (a) through (d), what is the lowest degree each function could have?

9-107. Consider the function $y=x^3-9x$.

a. What are the roots of the function? (Factoring will help!)

b. Sketch a graph of the function.

9-108. Make rough sketches of the graphs of each of the following polynomial functions. Be sure to label the x- and y- intercepts.

a. $y=x(2x+5)(2x-7)$

b. $y=(15-2x)^2(x+3)$

9-109. The management of the Carnival Cinema was worried about breaking even on their movie "Elvis Returns from Mars." To break even, they had to take in \$5000 on the matinee. They were selling adult tickets for \$8.50 and children's tickets for \$5.00. They knew they had sold a total of 685 tickets. How many of those tickets would have to have been adult tickets for them to meet their goal?

9-110. You are given the equation $5x^2+bx+20=0$. For what values of b does this equation have real solutions?

9-111. Show that each of the following equations is true.

a. $(i-3)^2=8-6i$

b. $(2i-1)(3i+1)=-7-i$

c. $(3-2i)(2i+3)=13$

9-112. Kahlid is going to make a table to graph $y=\sqrt{x^2}$, but Aaron says that it would be a waste of time to make a table because the graph is the same as $y=x$.

a. Use Kahlid's equation to make the table and draw the graph.

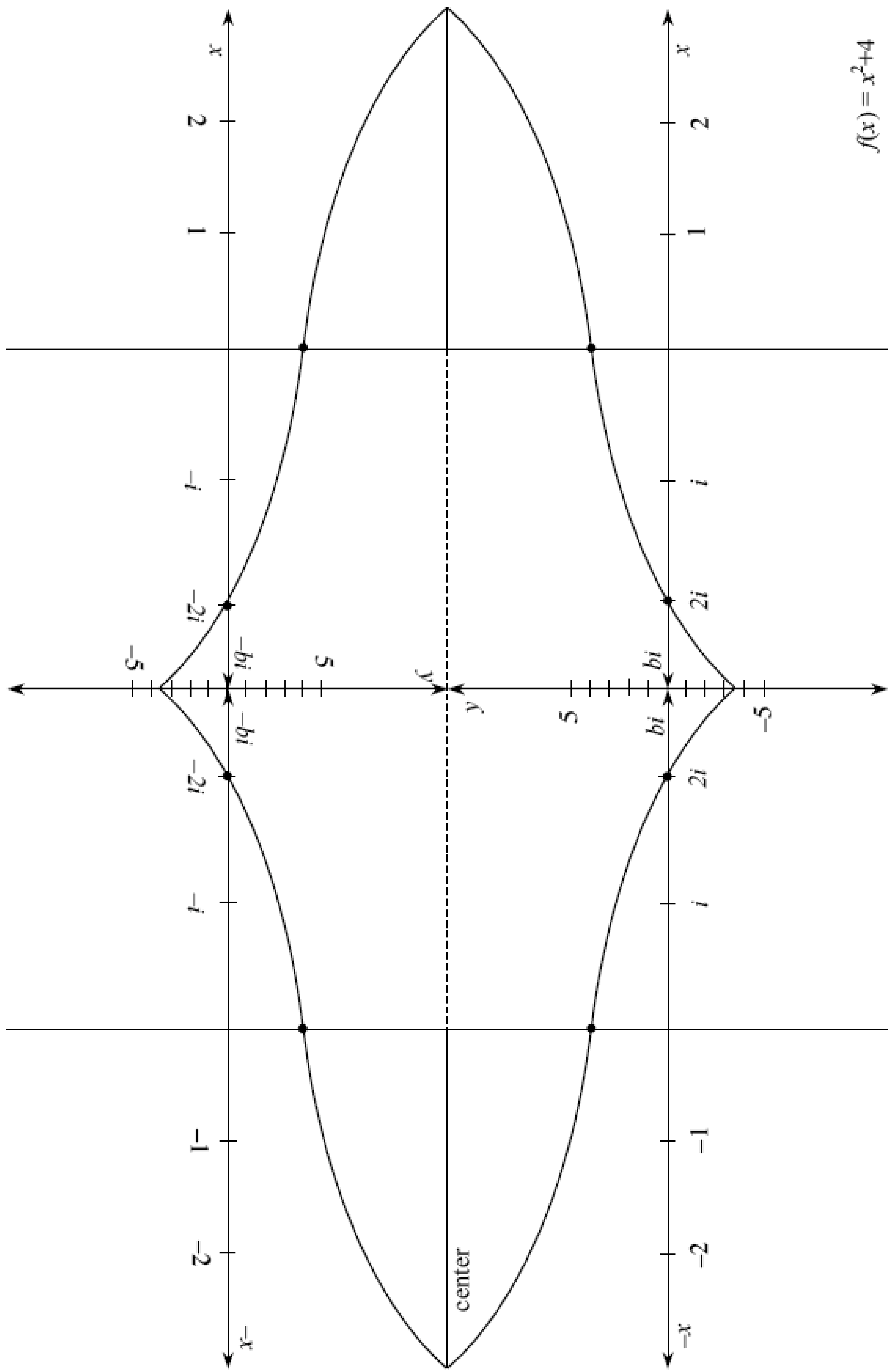
b. As you can see, Aaron was wrong, but you did get a graph that you have seen before. What other equation has the same graph? Explain why this is reasonable.

9-113. Multiply each of the following expressions.

a. $(x-3)^2$

b. $2(x+3)^2$

c. $(a-b)(a^2+ab+b^2)$


$f(x) = x^2+4$
center
x
−x
y
2
1
i
2i
bi
−i
−2i
−bi
5
−5
−1
−2

9.3.1 How can I divide polynomials?

Polynomial Division

When you graphed polynomial functions in the first section of this chapter, you learned that the factored form of a polynomial is very useful for finding the roots of the function or the x-intercepts of the graph. But what happens when you do not have the factored form and you need to find all of the roots?

9-114. Andre needs to find the exact roots of the function $f(x) = x^3 + 2x^2 - 7x - 2$. When he uses his graphing calculator, he can see that one of the x-intercepts is 2, but there are two other intercepts that he cannot see exactly. What does he need to be able to do to find the other roots?

Andre remembers that he learned how to multiply binomials and other polynomials using generic rectangles. He figures that since division is the inverse (or undo) operation for multiplication, he should be able to reverse the multiplication process to divide. As he thinks about that idea, he comes across the following news article.

Polydoku Craze Sweeping Nation!

(CPM) – Math enthusiasts around the nation have entered a new puzzle craze involving the multiplication of polynomials. The goal of the game, which enthusiasts have named Polydoku, is to fill in squares so that the multiplication of two polynomials will be completed.

	1	2	3	4	5
A	$\times$	$2x^3$	$-x^2$	$+3x$	-1
B	$3x$	$6x^4$	$-3x^3$	$9x^2$	$-3x$
C	-2	$-4x^3$	$2x^2$	$-6x$	2

$6x^4$ $-7x^3$ $+11x^2$ $-9x$ $+2$

The game shown at right, for example, represents the multiplication of $(3x-2)(2x^3 - x^2 + 3x - 1) = 6x^4 - 7x^3 + 11x^2 - 9x + 2$.

Most of the squares are blank at the start of the game. While the beginner level provides the factors (in the gray squares), some of the factors are missing in the more advanced levels.

9-115. Andre decided to join the craze and try some Polydoku puzzles, but he is not sure how to fill in some of the squares. Help him by answering parts (a) and (b) below about the Polydoku puzzle in the news article he read (found in problem 9-114), then complete part (c).

a. Explain how the term $2x^2$ in cell C3 of the news article was generated.

b. What values were combined to get $-7x^3$ in the news article answer?

c. Copy and complete the Polydoku puzzle at right.

	1	2	3	4	5
A	$\times$	$4x^3$	$+ 6x^2$	$- 2x$	$- 5$
B	$2x$				
C	$- 3$				

9-116. POLYDOKU TEAM CHALLENGE

Work with your team to complete the puzzle at right. Find the factors and the product for the puzzle. If you get stuck, you can consult parts (a) through (c) below for ideas.

	1	2	3	4	5
A	$\times$			$-2x$	
B	x	$2x^4$			
C	-4		$12x^2$		

$12x$

a. How is cell B2 related to the answer?

b. How did you find the third term in the answer?

c. What cells did you use to get the value in cell B5?

9-117. Jessica is about to start the intermediate-level Polydoku puzzle shown at right. Show Jessica how to complete the puzzle. Make sure you can **justify** your solution.

Use your results to complete the statements below.

	1	2	3	4
A	$\times$			
B	$2x$			
C	$+\ 5$			

$6x^3$ $+7x^2$ $-16x$ $+10$

$$\frac{6x^3+7x^2-16x+10}{2x+5} = \underline{\hspace{3cm}} \quad \text{and} \quad (2x+5)\cdot \underline{\hspace{3cm}} = \underline{\hspace{3.5cm}}$$

9-118. Unfortunately, Jessica made a mistake when she copied the problem. The constant term of the original polynomial was supposed to have the value + 18 (not + 10). She does not want to start all over again to solve the puzzle.

a. Jessica realizes that she now has 8 remaining from the original expression. What is the significance of this 8?

b. Jessica writes her work as shown below:

$$\frac{6x^3+7x^2-16x+18}{2x+5}=\frac{(6x^3+7x^2-16x+10)+8}{2x+5}=3x^2-4x+2 \text{ , remainder 8.}$$

Gina thinks that there is a way to write the answer without using the word "remainder." Discuss this with your team and find another way to write the result. Be prepared to share your results and your reasoning with the class.

c. Use Jessica and Gina's method to divide $(6x^3+11x^2-12x-1)\div(3x+1)$.

9-119. Create your own Polydoku puzzles that can be used to solve each of the polynomial-division problems below. Express any remainders as fractions and use your results to write a multiplication and a division statement such as those in problem 9-117.

a. $\dfrac{6x^4-5x^3+10x^2-18x+5}{3x-1}$

b. $(x^4-6x^3+18x-4)\div(x-2)$

c. $x-3\overline{\big)x^3+x^2-14x+3}$

d. $\dfrac{x^5-1}{x-1}$

9-120. Now work with your team to help Andre solve his original problem (problem 9-114). Find all of the roots (exact zeros) of the polynomial.

Math Notes

Methods and Meanings

Polynomial Division

The examples below show two methods for dividing $x^4 - 6x^3 + 18x - 1$ by $x - 2$. In both cases, the remainder is written as a fraction.

Using long division:

$$
\begin{array}{r}
x^3 - 4x^2 - 8x + 2 \\
x - 2\,\big)\,\overline{x^4 - 6x^3 + 0x^2 + 18x - 1} \\
\underline{x^4 - 2x^3} \qquad\qquad\qquad \\
-4x^3 + 0x^2 \qquad\qquad \\
\underline{-4x^3 + 8x^2} \qquad\qquad \\
-8x^2 + 18x \qquad \\
\underline{-8x^2 + 16x} \qquad \\
2x - 1 \\
\underline{2x - 4} \\
3
\end{array}
$$

Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

Using generic rectangles:

	x^3	$-4x^2$	$-8x$	$+2$	Remainder
x	x^4	$-4x^3$	$-8x^2$	$+2x$	3
-2	$-2x^3$	$+8x^2$	$+16x$	-4	

$x^4 - 6x^3 \quad + 0x^2 \quad + 18x \quad - 1$

Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

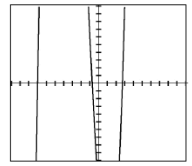
Therefore, $(x^4 - 6x^3 + 18x - 1) \div (x - 2) = x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$ and

$(x-2)(x^3 - 4x^2 - 8x + 2 + \frac{3}{x+2}) = x^4 - 6x^3 + 18x - 1$

May 01, 2011

Review & Preview

9-121. Carlos is always playing games with his graphing calculator, but now his calculator has contracted a virus. The TRACE, ZOOM, and WINDOW functions on his calculator are not working. He needs to solve $x^3+5x^2-16x-14=0$, so he graphs $y=x^3+5x^2-16x-14$ and sees the graph at right in the standard window.



a. From the graph, what appears to be an integer solution to the equation?

b. Check your answer from part (a) in the equation.

c. Since $x=-7$ is a solution to the equation, what is the factor associated with this solution?

d. Use polynomial division to find the other factor.

e. Use your new factor to complete this equation:

$$x^3+5x^2-16x-14=(x+7)(other\ factor)=0$$

f. The "other factor" leads to two other solutions to the equation. Find these two new solutions and give all three solutions to the original equation.

9-122. Now Carlos needs to solve $2x^3+3x^2-8x+3=0$, but his calculator will still only create a standard graph. He sees that the graph of $y=2x^3+3x^2-8x+3$ crosses the x-axis at $x=1$. Find all three solutions to the equation.

9-123. Without actually multiplying, decide which of the following polynomials could be the product of $(x-2)(x+3)(x-5)$. **Justify** your choice.

a. $x^3-4x^2-11x-5$ b. $2x^3-4x^2-11x+30$

c. $x^3-4x^2-11x+30$ d. $2x^3-4x^2-11x-5$

9-124. Which of the following binomials could be a factor of $x^3-9x^2+19x+5$? **Explain** your reasoning.

a. $x-2$ b. $x-5$ c. $x+3$ d. $x+2$

9-125. Now divide $x^3-9x^2+19x+5$ by the factor that you chose in the preceding problem. If it is a factor, use it and the resulting factor to find all the zeros of the polynomial. If it is not a factor, reconsider your answer to the preceding problem and try a different factor.

9-126. Consider the equation $5x^2-7x-6=0$ as you answer the questions in parts (a) through (d) below.

a. What are the factors of $5x^2-7x-6$?

b. What are the solutions to the equation?

c. **Explain** the relationship between the factors of the polynomial **expression** and the solutions to the equation.

d. How are the solutions to the equation related to the lead coefficient and constant term in the original polynomial?

9-127. This is a checkpoint for solving systems of three equations and three unknowns.

Use elimination or matrix multiplication to solve the system of equations at right.

$$x+y-z=12$$
$$3x+2y+z=6$$
$$2x+5y-z=10$$

Check your solution by referring to the Checkpoint 18 materials located at the back of your book.

If you needed help to solve this system correctly, then you need more practice solving 3×3 systems. Review the Checkpoint 18 materials and try the practice problems. Also, consider getting help outside of class time. From this point on, you will be expected to solve systems such as these accurately.

9-128. Given the equation: $3x+y-z=6$.

a. Draw a graph.

b. Is $(1, 2, -1)$ on the graph? **Justify** your answer.

9-129. Frank says that $\frac{a}{x+b}=\frac{a}{x}+\frac{a}{b}$, while Fred does not think that the two expressions are equivalent. Who is correct? **Justify** your answer.

9-130. The area of ΔABC is 24 square inches. If $\overline{AB}\perp\overline{BC}$ and $BC=8$ inches, find AC.

9.3.2 How can I solve it?

Factors and Integral Roots

You already know several methods for solving quadratic equations, such as factoring, completing the square, and using the Quadratic Formula. Mathematicians have developed formulas for solving third- and fourth-degree polynomial equations, but these formulas are far more complicated and messy than the Quadratic Formula is, and they are rarely used. Furthermore, for polynomials of degree greater than four, there is no single formula to use. For many polynomials, you can develop more useful methods than a formula based on what you already know.

9-131. SEARCHING FOR ROOTS OF POLYNOMIALS

By combining what you know about graphing, factoring, and polynomial division, and then applying what you know about solving quadratic equations, you will be able to find the roots of the higher-degree polynomial functions in this lesson.

Your task: Find all of the zeros of the polynomial below and then write the polynomial in factored form with factors of degree 2 or 1.

$$P(x) = x^4 - x^3 - 5x^2 + 3x + 6$$

Discussion Points

What are some possible linear factors?

How can the graph help us decide which factors to try?

How can we use the known factors to figure out other factors?

What do we need to do to write the polynomial in factored form and find the zeros?

Further Guidance

9-132. Using only integers for a, and just by looking at the polynomial expression, list all of the possible linear factors $(x \pm a)$ for the polynomial $P(x) = x^4 - x^3 - 5x^2 + 3x + 6$.

a. Could $x - 5$ be included on your list of possible linear factors? Explain.

b. Not all of the *possibilities* will actually be factors. Use your graphing calculator to decide which members of your list are the best possibilities.

c. Now that you have shortened the list of possibilities, which factors on the shortened list really are factors of $P(x)$? **Justify** your answer.

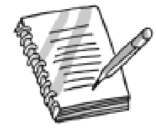
d. If you haven't already divided, divide the polynomial by one of the factors from part (c) and write the polynomial as a product of a linear and a cubic factor.

e. Now divide the cubic factor from part (d) by the other linear factor that worked, and write the original polynomial as a product of two linear factors and one quadratic factor.

f. From the factored form, you can find all of the solutions to $x^4 - x^3 - 5x^2 + 3x + 6 = 0$ and the exact x-intercepts for the graph of $P(x)$. What are they?

Further Guidance section ends here.

9-133. LEARNING LOG



As a team, look back over the work you did to find the linear factors of $P(x)$ and make a list of steps you can use to find all of the zeros of a given polynomial. Then record your ideas in your Learning Log. Use diagrams, arrows, and other math tools to help demonstrate your ideas. Label the list "Factors and Roots of Polynomial Functions."

9-134. Your teacher will assign your team one of the following polynomial functions. Use the list of steps your team developed to factor the polynomial and find all of its roots. Then prepare a poster in which you illustrate and **justify** each of your steps. Be sure to include the graph on your poster and clearly explain the relationship between the solutions of the equation and the x-intercepts.

a. $Q_1(x) = x^3 + 3x^2 + 1x - 5$

b. $Q_2(x) = 6x^4 + 7x^3 - 36x^2 - 7x + 6$

c. $Q_3(x) = x^4 + 2x^3 + 10x^2 + 18x + 9$

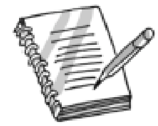
d. $Q_4(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

e. $Q_5(x) = x^5 - 4x^3 - x^2 + 4$

f. $Q_6(x) = x^3 + x^2 - 7x - 7$

9-135. LEARNING LOG

Make additions and adjustments to your Learning Log entry from problem 9-133 to reflect what you leaned from the posters.



9-136. Use the procedures you developed to factor each of the following polynomial expressions. Each final answer should be a linear factor times a quadratic factor. Look for patterns in the factors.

a. $x^3 - 1$ b. $x^3 + 8$ c. $x^3 - 27$ d. $x^3 + 125$

9-137. If you **generalize** the patterns for the factors in problem 9-136, you will discover a shortcut for factoring cubic polynomials that are described as the sum or difference of two cubes. Write the factors for each polynomial expression below.

a. $x^3 + a^3$

b. $x^3 - b^3$

c. Write a description of how to get the factors without having to divide.

9-138. Are there similar patterns for $x^4 + a^4$ and $x^4 - b^4$? Explain.

9-139. BUILDING POLYNOMIALS

For each of the following descriptions of polynomial functions with integral coefficients, answer each question below.

i. What are the possible numbers of real zeros?

ii. How many complex zeros are possible?

iii. For each number of possible real zeros, give an example of a polynomial in factored form.

a. A third-degree polynomial function.

b. A fourth-degree polynomial function.

c. A fifth-degree polynomial function.

MATH NOTES

METHODS AND MEANINGS

The Factor and Integral Zero Theorems

Factor Theorem: If a is a zero of a polynomial function, then $(x-a)$ is a factor of the polynomial, and if $(x-a)$ is a factor of the polynomial, then a is a zero.

Example: We know that -3, $\frac{1}{2}$, and $1 \pm i\sqrt{7}$ are zeros of the polynomial $p(x) = 2x^4 + x^3 + 22x^2 + 80x - 24$. (You can test each value by substituting it into the equation to verify that the resulting value of the function is 0.) According to the Factor Theorem, $(x+3)$, $(x-\frac{1}{2})$, $(x-(1+i\sqrt{7}))$, and $(x-(1-i\sqrt{7}))$ are factors of the polynomial. Notice that the theorem does not state that they are the *only* factors of the polynomial. In order to show that these are factors, we must show that they can each be part of a product that results in $p(x)$. We start by multiplying the factors together. To make the operations simpler, we multiply the complex factors first, as shown below.

$$\begin{aligned}&(x+3)(x-\tfrac{1}{2})(x-(1+i\sqrt{7}))(x-(1-i\sqrt{7}))\\&=(x+3)(x-\tfrac{1}{2})(x^2-2x+8)\end{aligned}$$

We then notice that if we multiply this expression by 2, we can eliminate the fraction in $(x-\frac{1}{2})$ without changing the roots. For convenience, we multiply $2(x-\frac{1}{2})$ first and then multiply the rest of the factors together.

$$\begin{aligned}&2(x+3)(x-\tfrac{1}{2})(x^2-2x+8)\\&=(x+3)(2x-1)(x^2-2x+8)\\&=2x^4+x^3+22x^2+80x-24=p(x)\end{aligned}$$

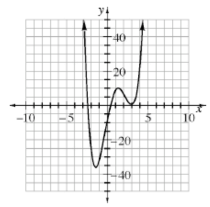
Integral Zero Theorem: For any polynomial with integral coefficients, if an integer is a zero of the polynomial, it must be a factor of the constant term.

Example: Suppose the integers a, b, c, and d are zeros of a polynomial. Then, according to the Factor Theorem, $(x-a)(x-b)(x-c)(x-d)$ are factors of the polynomial.

When you multiply these factors together, the constant term will be $abcd$, so a, b, c, and d are factors of the constant term.

Review & Preview

9-140. Carlo was trying to factor the polynomial $p(x) = x^4 - 4x^3 - 4x^2 + 24x - 9$ and find all of its roots. He had already found one factor by making a guess and trying it, so he had $p(x) = (x-3)(x^3 - x^2 - 7x + 3)$. He was trying to factor $x^3 - x^2 - 7x + 3$, and he had tried $(x+3)$, $(x+1)$, and $(x-1)$, but none worked. Then Teo came by and said, *"You should look at the graph."*



a. How does the graph help?

b. Complete the problem.

9-141. Spud has a problem. He knows that the solutions for a quadratic equation are $x = 3 + 4i$ and $x = 3 - 4i$, but in order to get credit for the problem he was supposed to have written down the equation. Unfortunately, he lost the paper with the original equation on it. His friends are full of advice.

a. Alexia says, *"Look, just remember when we made polynomials. If you wanted 7 and 4 to be the answers, you just used $(x-7)(x-4)$. So you just do x minus the first one times x minus the other."* Use $(x-(3+4i))(x-(3-4i))$ to find the quadratic expression.

b. Hugo says, *"No, no, no. You can do it that way, but that's too complicated. I think you just start with $x = 3 + 4i$ and work backward. So $x - 3 = 4i$, then, hmmm. Yeah, that'll work."* Try Hugo's method.

c. Which way do you think Spud should use? Explain your choice.

9-142. So far you have been able to extend the rules for real numbers to add, subtract, and multiply complex numbers, but what about dividing? Can you use what you know about real numbers to divide one complex number by another? In other words, if a problem looks like this:

$$\frac{3+2i}{-4+7i}$$

What needs to be done to get an answer in the form of a single complex number, $a + bi$?

Natalio had an idea. He said, *"I'll bet we can use the conjugate!"*

"How?" asked Ricki.

9-142. *Problem continued from previous page.*

"Well, it's a fraction. Can't we multiply the numerator and denominator by the same number?" Natalio replied.

a. Natalio was not very clear in his explanation. Show Ricki what he meant they should do.

b. Using Natalio's ideas you probably still came up with a fraction in part (a), but the denominator should be a whole number. To write a complex number such as $\frac{c+di}{m}$ in the form $a + bi$, just use the distributive property to rewrite the result as $\frac{c}{m} + \frac{d}{m}i$. Rewrite your result for part (a) in this form.

9-143. Use the method developed in problem 9-142 to do the following division problems.

a. $\frac{2-5i}{1-2i}$

b. $(-3+i) \div (2+3i)$

9-144. Find the inverse of $g(x) = (x+1)^2 - 3$ with the domain $x \geq -1$. Sketch both graphs and tell the domain and range of the inverse function.

9-145. Sketch the graph of each polynomial function below and find all of the zeros.

a. $y = x^3 + 1$

b. $y = x^3 - 8$

9-146. Solve the system of equations at right for (x, y, z).

$$x = y + z$$
$$2x + 3y + z = 17$$
$$z + 2y = 7$$

9-147. Sketch the graph of each equation below.

a. $y = 3\sin\left(x + \frac{\pi}{2}\right)$

b. $y = -2\sin(4x)$

9-148. Spud has done it again. He's lost another polynomial function. This one was a cubic, written in standard form. He knows that there were two complex zeros, $-2 \pm 5i$ and one real zero, -1. What could his original function have been?

9-149. Given the equation $x^3 - 6x^2 + 7x + 2 = 0$.

a. Verify that $x = 2$ is a solution.

b. What is one factor of $x^3 - 6x^2 + 7x + 2$?

c. Use (b) to find another factor.

d. What are all the solutions of $x^3 - 6x^2 + 7x + 2 = 0$?

9-150. Rewrite each of the following division problems as a single complex number.

a. $\frac{5+3i}{5-3i}$ b. $\frac{7+3i}{1-i}$

9-151. Where do the graphs below intersect? You should be able to do these without a graphing calculator.

a. $2x + y = 10$
$x + y = 25$

b. $2x + y = 10$
$x^2 + y^2 = 25$

9-152. Sketch both the circle $x^2 + y^2 = 25$ and the parabola $y = x^2 - 13$.

a. How many points of intersection are there?

b. Find the coordinates of these points algebraically.

9-153. Solve each equation. Be sure to check your answers.

a. $\sqrt{x} + 2 = x$ b. $\sqrt{x} + 2 = \sqrt{x+6}$

9-154. Find the distance between each pair of points.

a. (2, 3) and (4, 7) b. (2, 3) and (x, y)

9-155. For each equation, find two solutions $0 \le x < 2\pi$, which make the equation true. No calculator necessary.

a. $\cos x = -\frac{1}{2}$ **b.** $\tan x = \frac{\sqrt{3}}{3}$ **c.** $\sin x = 0$ **d.** $\cos x = \frac{\sqrt{2}}{2}$

9-156. Rewrite each of the following division problems as a single complex number in simplest form.

a. $\frac{2-6i}{4+2i}$ b. $\frac{5}{1+2i}$

9-157. A long lost relative died and left you \$15,000! Your parents say that you need to save the money for college, so you put it an account that pays 8% interest compounded annually. How many years will it take until your account is worth \$25,000?

9-158. Solve each equation.

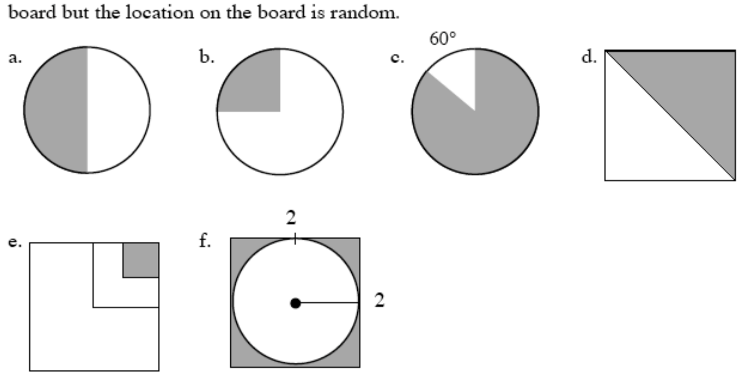
a. $\log_3(2x-1) = -2$ b. $5^{\log_5(x)} = 3$

c. $\log_2(x) - \log_2(3) = 4$ d. $\log_3(5) = x$

9-159. Verify that the graphs of the equations $x^2 + y^3 = 17$ and $x^4 - 4y^2 - 8xy = 17$ intersect at $(3, 2)$.

9-160. The graphs of $y = \log_2(x-1)$ and $y = x^3 - 4x$ intersect at two points: $(2, 0)$ and approximately $(1.1187, -3.075)$. Use that information to solve $\log_2(x-1) = x^3 - 4x$.

9-161. Each dartboard below is a target at the county fair dart-throwing game. What is the probability of hitting the darkened region of each target? Assume you always hit the board but the location on the board is random.



9-162. Use your knowledge of the unit circle to explain why the graphs of $y = \sin\theta$ and $y = \cos(\theta - \frac{\pi}{2})$ are the same.

9-163. For homework, Londa was asked to simplify the expression $\sqrt{-7} \cdot \sqrt{-7} =$. She got the answer 7, but when she checked, she learned that the correct answer was -7.

a. Show Londa the steps she could take to get -7.

b. What steps do you think Londa took to get 7 as a result?

c. What does she need to consider in order to avoid making this mistake in the future?

d. Londa's example means that it is not always true that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ for real numbers a and b. What restriction needs to be placed on the numbers a and b?

9-164. Change each angle from degrees to radians.

a. 60° b. 75° c. 210° d. 225°

9.3.3 How can I use it?

An Application of Polynomials

In this lesson, you will have the opportunity to use the equation and graph of a polynomial function to solve a problem involving a game at the county fair.

9-165. COUNTY FAIR GAME TANK

The Mathamericaland Carnival Company wants to create a new game. It will consist of a tank filled with ping-pong balls of different colors. People will pay for the opportunity to crawl around in the tank blindfolded for 60 seconds, while they collect ping-pong balls. Most of the ping-pong balls will be white, but there will be a few of different colors. The players will win \$100 for each red, \$200 for each blue, and \$500 for each green ping-pong ball they carry out of the tank.

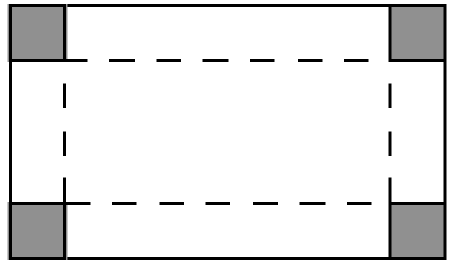
The tank will be rectangular, open at the top, and will be made by cutting squares out of each corner of an 8.5-meter by 11-meter sheet of translucent miracle material that can be bent into shape. The owner of the company thinks that she will make the greatest profit if the tank has maximum volume. She has hired your team to figure out the exact dimensions of the tank that creates the maximum value.

Your task: Your team will write a report of your findings to the Carnival Company that includes each of the following elements.

- Any data or conjectures your team made based on experimental paper tanks.
- A diagram of the tank with the dimensions clearly labeled with appropriate variables.
- A graph of the volume function that you found with notes on a reasonable domain and range.
- An equation that matches your graph.
- Your conclusions and observations.

Further Guidance

9-166. Use a full sheet of 8.5" x 11" paper, which is the same shape as the material for the tank. Each member of your team should choose a different sized square to cut out of the corners. Measure the side of the square you cut out and write along the edge of the large piece of paper. Use either inches or centimeters. Your paper should look like the figure at right.



Fold the paper up into an open box (fold on the dotted lines). Then tape the cut parts together so that the box holds its shape. Measure the dimensions of the tank. Record the dimensions directly on the model tank.

9-167. Make a table like the one below for your team's results. Consider "extreme" tanks, the ones with the largest possible cutout and the smallest possible cutout. For example, imagine cutting a square out of each corner zero inches on a side.

Height of Tank	Width of Tank	Length of Tank	Volume of Tank

a. Examine the data in the table with your team and make some conjectures about how to find the maximum volume.

b. Label the height as x. Using x for the height, find expressions for the length and width.

c. Write an equation to represent the volume of the tank.

d. Sketch the graph of your function by using the roots and determining the orientation.

e. What domain and range make sense for your function?

f. Approximate the maximum volume of the tank and the dimensions of the tank that will generate this volume.

9-168. Use your graph and your tank model to write your report. Include your answers to the following questions.

a. Which points on the graph represent tanks that can actually be made? Explain.

b. How are the dimensions of the tank related? In other words, what happens to the length and width as the height increases?

c. Make a drawing of your tank. (You may want to use isometric dot paper.) Label your drawing with its dimensions and its volume.

Further Guidance section ends here.

METHODS AND MEANINGS

MATH NOTES

Factoring Sums and Differences

The difference of two squares can be factored: $a^2 - b^2 = (a+b)(a-b)$

The sum of two cubes can be factored: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

The difference of two cubes can be factored: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Review & Preview

9-169. A polynomial function has the equation $P(x) = x(x-3)^2(2x+1)$. What are the x-intercepts?

9-170. Sketch a graph of a fourth-degree polynomial that has no real roots.

9-171. Generally, when you are asked to factor, it is understood that you are only to use integers in your factors. If you are allowed to use irrational or complex number, *any* quadratic can be factored.

By setting the polynomial equal to zero and solving the quadratic equation, you can work backwards to "force factor" any quadratic. Use the solutions of the corresponding quadratic equation to write each of the following expressions as a product of two linear factors.

a. $x^2 - 10$ b. $x^2 - 3x - 7$ c. $x^2 + 4$ d. $x^2 - 2x + 2$

9-172. Sketch the graphs and find the area of the intersection of the inequalities below.

$$y > |x+3|$$
$$y \le 5$$

9-173. Determine whether $x = -2$ is a solution to the equation $x^4 - 4x = 8x^2 - 40$. Show why or why not.

9-174. Use your solving skills to complete parts (a) and (b) below.

a. Solve $\frac{x+3}{x-1} - \frac{x}{x+1} = \frac{8}{x^2-1}$ for x.

b. In part (a), the result of solving the equation is $x = 1$, but what happens when you substitute 1 for x? What does this mean in relation to the solutions for this equation?

9-175. The roots of two quadratic polynomials are given below. Write possible quadratic functions in standard form.

a. $x = -i,\ x = i$ b. $x = 1 + \sqrt{2},\ x = 1 - \sqrt{2}$

9-176. Graph two cycles of each function.

a. $y = -2\cos(x + \frac{\pi}{2})$ b. $y = \sin(x - \frac{\pi}{2})$

CL 9-177. Decide if each of the following equations is a polynomial. If it is, state the degree. If it is not, explain how you know.

a. $f(x) = 3x^3 - 2x + 5$

b. $y = 0.25x^7 - 5x$

c. $y = 3^x - x^2$

d. $f(x) = x^2 - \sqrt{x} + 2$

e. $Q(x) = 3(x-4)^2(x+2)$

f. $y = x^2 - 3x + 5 - \frac{2}{x-2}$

CL 9-178. Where do the graphs of each of the following functions cross the x-axis?

a. $f(x) = (x-2)^2 - 3$

b. $f(x) = (x-19)^2(x+14)$

CL 9-179. Write a polynomial equation for a graph that has three x-intercepts at $(-3, 0)$, $(2, 0)$, and $(5, 0)$, and passes through the point $(1, 56)$.

CL 9-180. Decide if each of the following functions has real or complex roots.

a. $y = 3x^2 + 5x + 4$

b. $y = 3x^2 + 5x - 4$

CL 9-181. Make a sketch of a graph $q(x)$ such that $q(x) = 0$ would have the number and type of solutions indicated below.

a. 7 real solutions

b. 5 real and 2 complex solutions

c. 4 complex solutions

d. 2 complex and 4 real solutions

CL 9-182. Sketch graphs of each of the following polynomial functions. Be sure to label the x- and the y-intercepts of each graph.

a. $y = x(2x+3)(2x-5)$

b. $y = (11-2x)^2(x-2)$

CL 9-183. Simplify each expression

a. $(3+4i)+(7-2i)$

b. $(3+5i)^2$

c. $(7+i)(7-i)$

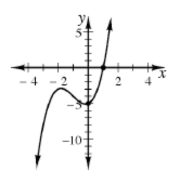
d. $(3i)(2i)^2$

e. i^3

f. i^{32}

CL 9-184. Divide: $(2x^3 + x^2 - 19x + 36) \div (x+4)$

CL 9-185. The graph of $f(x) = x^3 + 3x^2 + x - 5$ is shown at right. Use it to determine all real and complex roots.



CL 9-186. The roots of a quadratic polynomial are given below. Find a possible quadratic equation in standard form.

a. $x = 2i,\ x = -2i$

b. $x = 2+\sqrt{3},\ x = 2-\sqrt{3}$

CL 9-187. Use the system at right to answer each of the following questions.

$y = 2x$
$y = x^2 + 5$

a. Without graphing, what is the solution to the system?

b. What does the solution to the system tell you about the graphs?

CL 9-188. Check your answers using the table at the end of this section. Which problems do you feel confident about? Which problems were hard? Have you worked on problems like these in math classes you have taken before? Use the table to make a list of topics you need help on and a list of topics you need to practice more.